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Nonlinear Ergodic Theorems for Semigroups of Nonexpansive Mappings and Left Ideals

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1 Introduction

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $s \in S$ the mappings $s \rightarrow a \cdot s$ and $s \rightarrow s \cdot a$ from S to S are continuous. Let E be a uniformly convex Banach space and let $S = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings on a closed convex subset C of E into C , i.e., $T_{ab}x = T_aT_bx$ for every $a, b \in S$ and $x \in C$ and the mapping $(s, x) \rightarrow T_s(x)$ from $S \times C$ into C is continuous when $S \times C$ has the product topology. Let $F(S)$ denote the set $\{x \in C : T_sx = x \text{ for all } s \in S\}$ of common fixed points of S in C . Then as well known, $F(S)$ (possibly empty) is a closed convex subset of C (see [5]).

In this paper, we shall study the distance between left ideal orbits and elements in the fixed point set $F(S)$. We shall prove (Theorem 3.11) among other things that if E has a Fréchet differentiable norm, then for any semitopological semigroup S and $x \in C$, the set $Q(x) = \bigcap \overline{\text{co}}\{T_t x : t \in L\}$, with the intersection taking over all closed left ideals L of S , contains at most one common fixed point of S (where $\overline{\text{co}}A$ denotes the closed convex hull of A). This result is then applied to show (Theorem 4.1) that if $F(S) \cap Q(x) \neq \emptyset$ for any $x \in C$, then there exists a retraction P from C onto $F(S)$ such that $T_t P = P T_t = P$ for every $t \in S$ and $P(x) \in \overline{\text{co}}\{T_t x : t \in S\}$ for every $x \in C$. Both Theorem 3.11 and Theorem 4.1 were established by Lau and Takahashi in [18] when S has finite intersection property for closed left ideals.

The first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Baillon [1]: Let C be a closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{\text{co}}\{T^n x : n = 1, 2, \dots\}$ for each $x \in C$. In [24], Takahashi proved the existence of such a retraction for an amenable semigroup. This result is further extended to certain Banach spaces by Hirano and Takahashi in [12].

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Our paper is organized as follows : In section 2 we define some terminologies that we use ; in section 3 we study the distance between ideals determined by left orbits and the fixed point set ; in section 4 we apply our results in section 3 to establish our main nonlinear ergodic theorems ; finally in section 5 we study an almost fixed point property determined by the minimal left ideals in the enveloping semigroup of a semigroup of nonexpansive mappings on a weak compact convex set and obtain a generalization of De Marr's fixed point theorem [6].

2 Preliminaries

Throughout this paper, we assume that a Banach (or Hilbert) space is real.

Let E be a Banach space and let E^* be its dual. Then, the value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f \rangle$ or $f(x)$. The duality mapping J of E is a multivalued operator $J : E \rightarrow E^*$ where $J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$ (which is nonempty by simple application of the Hahn-Banach theorem). Let $B = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . Then the norm of E is said to be Fréchet differentiable if for each $x \in B$, the limit

$$\lim_{\lambda \rightarrow 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}$$

is attained uniformly for $y \in B$. In this case, J is a single-valued and norm to norm continuous mapping from E into E^* (see [5] or [8] for more details).

Let S be a nonempty set and let X be a subspace of $l^\infty(S)$ (bounded real-valued functions on S) containing constants. By a submean on X we shall mean a real-valued function μ on X satisfying the following properties:

- (1) $\mu(f + g) \leq \mu(f) + \mu(g)$ for every $f, g \in X$;
- (2) $\mu(\alpha f) = \alpha \mu(f)$ for every $f \in X$ and $\alpha \geq 0$;
- (3) For $f, g \in X$, $f \leq g$ implies $\mu(f) \leq \mu(g)$;
- (4) $\mu(c) = c$ for every constant function c .

A semitopological semigroup S is called left reversible (resp. right reversible) if S has finite intersection property for right (resp. left) ideals. S is called reversible if S is both left and right reversible.

Let S be a semitopological semigroup and let $C(S)$ denote the closed subalgebra of $l^\infty(S)$ consisting of bounded continuous functions. For each $f \in C(S)$ and $a \in S$, let $(l_a f)(t) = f(at)$ and $(r_a f)(t) = f(ta)$. Let $RUC(S)$ denote all $f \in C(S)$ such that the mapping $S \rightarrow C(S)$ defined by $s \rightarrow r_s f$ is continuous when $C(S)$ has the norm topology. Then $RUC(S)$ is a translation invariant subalgebra of $C(S)$ containing constants. Further, $RUC(S)$ is precisely the space of bounded left uniformly continuous functions on S when S is a group (see [11]).

A submean μ on $RUC(S)$ is called invariant if $\mu(l_a f) = \mu(r_a f) = \mu(f)$ for every $f \in RUC(S)$ and $a \in S$. If S is a discrete semigroup, then $RUC(S)$ has an invariant submean if and only if S is reversible. Also if S is normal and $C(S)$ has an invariant submean, then S is reversible. However S need not be reversible when $C(S)$ has an invariant submean in general (see [19] for details).

3 Left ideal orbits and the fixed point set

Unless otherwise specified, S denotes a semitopological semigroup and $\mathcal{S} = \{T_s : s \in S\}$ a continuous representation of S as nonexpansive mappings from a nonempty closed convex subset C of a Banach space E into C .

Let $\mathcal{L}(S)$ denote the collection of closed left ideals in S . Assume that $F(S) \neq \emptyset$. For each $x \in C$ and $L \in \mathcal{L}(S)$, define the real-valued function $q_{x,L}$ on $F(S)$ by

$$q_{x,L}(f) = \inf\{\|T_t x - f\|^2 : t \in L\}$$

and let

$$q_x(f) = \sup\{q_{x,L} : L \in \mathcal{L}(S)\}.$$

Then

$$q_x(f) = \sup_s \inf_t \|T_{ts} x - f\|^2$$

as readily checked.

LEMMA 3.1 *Let C be a nonempty closed convex subset of a Banach space E . If $F(S) \neq \emptyset$, then for each $x \in C$, q_x is a continuous real-valued function on $F(S)$ such that $0 \leq q_x(f) \leq \|x - f\|^2$ for each $f \in F(S)$ and $q_x(f_n) \rightarrow \infty$ if $\|f_n\| \rightarrow \infty$. Further, if $F(S)$ is convex, then q_x is a convex function on $F(S)$.*

Proof. Since $0 \leq \|T_t x - f\|^2 = \|T_t x - T_t f\|^2 \leq \|x - f\|^2$ for every $f \in F(S)$ and $t \in S$, it follows readily that $0 \leq q_x(f) \leq \|x - f\|^2$. Also if $f \in F(S)$ and $t \in S$, then $\|T_t x - f\| \leq \|x - f\|$. Hence $\|T_t x\| \leq \|T_t x - f\| + \|f\| \leq \|x - f\| + \|f\|$, i.e., $M = \sup\{\|T_t x\| : t \in S\} < \infty$. Let $\{f_n\}$ be a sequence in $F(S)$ such that $\|f_n\| \rightarrow \infty$. Then we have for each $t \in S$,

$$\begin{aligned} \|T_t x - f_n\|^2 &\geq (\|T_t x\| - \|f_n\|)^2 \\ &= \|f_n\|^2 - 2\|T_t x\|\|f_n\| + \|T_t x\|^2 \\ &\geq \|f_n\|^2 - 2M\|f_n\| \\ &= \|f_n\|^2 \left(1 - \frac{2M}{\|f_n\|}\right) \end{aligned}$$

and hence for each $L \in \mathcal{L}(S)$,

$$q_{x,L}(f_n) \geq \|f_n\|^2 \left(1 - \frac{2M}{\|f_n\|}\right) \rightarrow \infty.$$

So we have $q_x(f_n) \rightarrow \infty$.

To see that q_x is continuous, let $\{f_n\}$ be a sequence in $F(S)$ converging to some $f \in F(S)$ and

$$M' = \sup\{\|T_t x - f_n\| + \|T_t x - f\| : n = 1, 2, \dots \text{ and } t \in S\}.$$

Then since

$$\begin{aligned} \|T_t x - f_n\|^2 - \|T_t x - f\|^2 &\leq (\|T_t x - f_n\| + \|T_t x - f\|) |\|T_t x - f_n\| - \|T_t x - f\|| \\ &\leq M' \|f_n - f\|, \end{aligned}$$

we have for each $L \in \mathcal{L}(S)$,

$$q_{x,L}(f_n) \leq q_{x,L}(f) + M'\|f_n - f\|.$$

Similarly, we have

$$q_{x,L}(f) \leq q_{x,L}(f_n) + M'\|f_n - f\|.$$

So we obtain

$$|q_x(f_n) - q_x(f)| \leq M'\|f_n - f\|.$$

This implies that q_x is continuous on $F(S)$.

If $F(S)$ is convex, for each $f, g \in F(S)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, $\alpha f + \beta g \in F(S)$. Let $\varepsilon > 0$. Then there exists $L_0 \in \mathcal{L}(S)$ such that

$$\sup_{L \in \mathcal{L}(S)} \inf_{t \in L} (\alpha \|T_t x - f\|^2 + \beta \|T_t x - g\|^2) < \inf_{t \in L_0} (\alpha \|T_t x - f\|^2 + \beta \|T_t x - g\|^2) + \frac{\varepsilon}{2}.$$

Let $u \in L_0$. Then $Su \subseteq L_0$ and hence

$$\sup_{L \in \mathcal{L}(S)} \inf_{t \in L} (\alpha \|T_t x - f\|^2 + \beta \|T_t x - g\|^2) < \inf_{t \in S} (\alpha \|T_{tu} x - f\|^2 + \beta \|T_{tu} x - g\|^2) + \frac{\varepsilon}{2}.$$

Moreover, there exist $v, w \in S$ such that

$$\|T_{vu} x - f\|^2 < \inf_{t \in S} \|T_{tu} x - f\|^2 + \frac{\varepsilon}{2}$$

and

$$\|T_{wvu} x - f\|^2 < \inf_{t \in S} \|T_{tvu} x - f\|^2 + \frac{\varepsilon}{2}.$$

Therefore we obtain

$$\begin{aligned} q_x(\alpha f + \beta g) &= \sup_{L \in \mathcal{L}(S)} \inf_{t \in L} \|T_t x - (\alpha f + \beta g)\|^2 \\ &\leq \sup_{L \in \mathcal{L}(S)} \inf_{t \in L} (\alpha \|T_t x - f\|^2 + \beta \|T_t x - g\|^2) \\ &< \inf_{t \in S} (\alpha \|T_{tu} x - f\|^2 + \beta \|T_{tu} x - g\|^2) + \frac{\varepsilon}{2} \\ &\leq \alpha \|T_{wvu} x - f\|^2 + \beta \|T_{wvu} x - g\|^2 + \frac{\varepsilon}{2} \\ &\leq \alpha \|T_{vu} x - f\|^2 + \beta \|T_{vu} x - g\|^2 + \frac{\varepsilon}{2} \\ &< \alpha \inf_{t \in S} \|T_{tu} x - f\|^2 + \beta \inf_{t \in S} \|T_{tvu} x - g\|^2 + \frac{\alpha \varepsilon}{2} + \frac{\beta \varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \alpha \inf_{t \in L_1} \|T_t x - f\|^2 + \beta \inf_{t \in L_2} \|T_t x - g\|^2 + \varepsilon \\ &\quad (\text{ where } L_1 = \overline{Su} \text{ and } L_2 = \overline{Svu}) \\ &\leq \alpha q_x(f) + \beta q_x(g) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$q_x(\alpha f + \beta g) \leq \alpha q_x(f) + \beta q_x(g). \square$$

THEOREM 3.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Assume that $F(S) \neq \emptyset$. Then for any $x \in C$, there exists a unique element $h \in F(S)$ such that*

$$q_x(h) = \inf\{q_x(f) : f \in F(S)\}.$$

Proof. Since E is uniformly convex, the fixed point set $F(S)$ in C is closed and convex (see [5]). Hence it follows from Lemma 3.1 and [2] that there exists $h \in F(S)$ such that

$$q_x(h) = \inf\{q_x(f) : f \in F(S)\}.$$

To see that h is unique, let $k \in F(S)$. Then by [27], there exists a strictly increasing and convex function (depending on h and k) $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and

$$\begin{aligned} \|T_t x - (\lambda h + (1 - \lambda)k)\|^2 &= \|\lambda(T_t x - h) + (1 - \lambda)(T_t x - k)\|^2 \\ &\leq \lambda\|T_t x - h\|^2 + (1 - \lambda)\|T_t x - k\|^2 - \lambda(1 - \lambda)g(\|h - k\|) \end{aligned}$$

for each $t \in S$ and λ with $0 \leq \lambda \leq 1$. So we have for each λ with $0 \leq \lambda \leq 1$,

$$\begin{aligned} q_x(h) &\leq q_x(\lambda h + (1 - \lambda)k) \\ &\leq \lambda q_x(h) + (1 - \lambda)q_x(k) - \lambda(1 - \lambda)g(\|h - k\|) \end{aligned}$$

and hence

$$q_x(h) \leq q_x(k) - \lambda g(\|h - k\|).$$

It follows that

$$q_x(h) \leq q_x(k) - g(\|h - k\|) \text{ as } \lambda \rightarrow 1.$$

Since g is strictly increasing, it follows that if $q_x(h) = q_x(k)$, then $h = k$. \square

We call the unique element $h \in F(S)$ in Theorem 3.2 the minimizer of q_x in $F(S)$. For each $x \in C$, let

$$Q(x) = \bigcap_{L \in \mathcal{L}(S)} \overline{\text{co}}\{T_t x : t \in L\} (= \bigcap_{s \in S} \overline{\text{co}}\{T_{ts} x : t \in S\}).$$

THEOREM 3.3 *Let C be a nonempty closed convex subset of a Hilbert space H . Let $S = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings from C into C . Then for any $x \in C$, any element in $Q(x) \cap F(S)$ is the unique minimizer of q_x in $F(S)$. In particular, $Q(x) \cap F(S)$ contains at most one point.*

Proof. Let $z \in F(S)$ be the minimizer of q_x in $F(S)$ and $y \in Q(x) \cap F(S)$. Then for some $\varepsilon > 0$, there exists $u \in S$ such that

$$\begin{aligned} \sup_s \inf_t (\|T_{ts} x - z\|^2 + 2\langle T_{ts} x - z, z - y \rangle + \|z - y\|^2) \\ < \inf_t (\|T_{tu} x - z\|^2 + 2\langle T_{tu} x - z, z - y \rangle + \|z - y\|^2) + \frac{\varepsilon}{4}. \end{aligned}$$

Moreover there exist $v, w \in S$ such that

$$\|T_{vu} x - z\|^2 < \inf_t \|T_{tu} x - z\|^2 + \frac{\varepsilon}{4}$$

and

$$\langle T_{wvu}x - z, z - y \rangle < \inf_t \langle T_{tvu}x - z, z - y \rangle + \frac{\varepsilon}{4}.$$

Therefore we obtain

$$\begin{aligned} q_x(y) &= \sup_s \inf_t \|T_{ts}x - y\|^2 \\ &= \sup_s \inf_t (\|T_{ts}x - z\|^2 + 2\langle T_{ts}x - z, z - y \rangle + \|z - y\|^2) \\ &< \inf_t (\|T_{tu}x - z\|^2 + 2\langle T_{tu}x - z, z - y \rangle + \|z - y\|^2) + \frac{\varepsilon}{4} \\ &\leq \|T_{wvu}x - z\|^2 + 2\langle T_{wvu}x - z, z - y \rangle + \|z - y\|^2 + \frac{\varepsilon}{4} \\ &\leq \|T_{vu}x - z\|^2 + 2\langle T_{wvu}x - z, z - y \rangle + \|z - y\|^2 + \frac{\varepsilon}{4} \\ &< \inf_t \|T_{tu}x - z\|^2 + 2\inf_t \langle T_{tvu}x - z, z - y \rangle \\ &\quad + \|z - y\|^2 + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \\ &\leq \sup_s \inf_t \|T_{ts}x - z\|^2 + 2\sup_s \inf_t \langle T_{ts}x - z, z - y \rangle \\ &\quad + \|z - y\|^2 + \varepsilon \\ &= q_x(z) + 2\sup_s \inf_t \langle T_{ts}x - z, z - y \rangle + \|z - y\|^2 + \varepsilon. \end{aligned}$$

This implies

$$\begin{aligned} 2\sup_s \inf_t \langle T_{ts}x - z, z - y \rangle &> q_x(y) - q_x(z) - \|z - y\|^2 - \varepsilon \\ &\geq -\|z - y\|^2 - \varepsilon. \end{aligned}$$

So, there exists $a \in S$ such that

$$2\langle T_{ta}x - z, z - y \rangle > -\|z - y\|^2 - \varepsilon$$

for every $t \in S$. From $y \in \overline{co}\{T_{ta}x : t \in S\}$, we have

$$2\langle y - z, z - y \rangle \geq -\|z - y\|^2 - \varepsilon.$$

This inequality implies $\|z - y\|^2 \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $z = y$. \square

REMARK 3.4 From Theorem 3.3, it is natural to ask the following:

Problem 1. If E is a uniformly convex Banach space, $x \in C$ and $y \in Q(x) \cap F(S)$, is y always the minimizer of q_x in $F(S)$?

Problem 2. If E is a uniformly convex Banach space, does $Q(x) \cap F(S)$ contain at most one point for each $x \in C$?

Clearly, by Theorem 3.2, an affirmative answer for Problem 1 gives an affirmative answer to Problem 2. We now proceed to give an affirmative answer for Problem 2 when E has a Fréchet differentiable norm.

LEMMA 3.5 Let C be a nonempty closed convex subset of a Banach space E . Let $x \in C$ and $f \in F(S)$. Then

$$\inf_s \|T_s x - f\| = \inf_s \sup_t \|T_{ts} x - f\|.$$

Proof. Let $r = \inf_s \|T_s x - f\|$ and $\varepsilon > 0$. Then there exists $a \in S$ such that

$$\|T_a x - f\| < r + \varepsilon.$$

So, for each $t \in S$, we have

$$\|T_{ta} x - f\| \leq \|T_a x - f\| < r + \varepsilon$$

and hence

$$\begin{aligned} \inf_s \sup_t \|T_{ts} x - f\| &\leq \sup_t \|T_{ta} x - f\| \\ &\leq r + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\inf_s \sup_t \|T_{ts} x - f\| \leq r.$$

It is clear that $\inf_s \sup_t \|T_{ts} x - f\| \geq r$. So we have

$$\inf_s \sup_t \|T_{ts} x - f\| = r. \square$$

LEMMA 3.6 Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $x \in C, f \in F(S)$ and $0 < \alpha \leq \beta < 1$. Then for any $\varepsilon > 0$, there exists a closed left ideal L of S such that

$$\|T_s(\lambda T_t x + (1 - \lambda)f) - (\lambda T_s T_t x + (1 - \lambda)f)\| < \varepsilon$$

for every $s \in S, t \in L$ and $\alpha \leq \lambda \leq \beta$.

Proof. Let $r = \inf_s \|T_s x - f\|$. By Lemma 3.5, for any $d > 0$, there exists $t_0 \in S$ such that

$$\sup_t \|T_{tt_0} x - f\| \leq r + d.$$

Apply now Lemma 1 in [18] and let $L = \overline{S t_0}$. \square

Let E be a Banach space and let S be a semigroup. Let $\{x_\alpha : \alpha \in S\}$ be a subset of E and $x, y \in E$. Then we write $x_\alpha \rightarrow x$ ($\alpha \rightarrow \infty_R$) if for any $\varepsilon > 0$, there exists $\alpha_0 \in S$ such that $\|x_{\alpha\alpha_0} - x\| < \varepsilon$ for every $\alpha \in S$ (see [23]). We also denote by $[x, y]$ the set $\{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1\}$.

LEMMA 3.7 Let C be a nonempty closed convex subset of a Banach space E with a Fréchet differentiable norm and let S be a semigroup. Let $\{x_\alpha : \alpha \in S\}$ be a bounded subset of C . Let $z \in \bigcap_\beta \overline{C\alpha}\{x_{\alpha\beta} : \alpha \in S\}$, $y \in C$ and $\{y_\alpha : \alpha \in S\}$ be a subset of C with $y_\alpha \in [y, x_\alpha]$ and

$$\|y_\alpha - z\| = \min\{\|u - z\| : u \in [y, x_\alpha]\}.$$

If $y_\alpha \rightarrow y$ ($\alpha \rightarrow \infty_R$), then $y = z$.

Proof. Since the duality mapping J of E is single-valued, for each $\alpha \in S$, it follows from [7] that

$$\langle u - y_\alpha, J(y_\alpha - z) \rangle \geq 0$$

for every $u \in [y, x_\alpha]$. Putting $u = x_\alpha$, we have

$$\langle x_\alpha - y_\alpha, J(y_\alpha - z) \rangle \geq 0$$

for every $\alpha \in S$. Since $\{x_\alpha : \alpha \in S\}$ is bounded, there exists $K > 0$ such that $\|x_\alpha - y\| \leq K$ and $\|y_\alpha - z\| \leq K$ for every $\alpha \in S$. Let $\varepsilon > 0$ and choose $\delta > 0$ so small that $2\delta K < \varepsilon$. Then since the norm of E is Fréchet differentiable, there exists $\delta_0 > 0$ such that $\delta_0 < \delta$ and

$$\|J(u) - J(y - z)\| < \delta$$

for every $u \in E$ with $\|u - (y - z)\| < \delta_0$. Since $y_\alpha \rightarrow y$ ($\alpha \rightarrow \infty_R$), there exists $\alpha_0 \in S$ such that

$$\|y_{\alpha_0} - y\| < \delta_0$$

for every $\alpha \in S$. So, for each $\alpha \in S$, we have

$$\begin{aligned} & |\langle x_{\alpha_0} - y_{\alpha_0}, J(y_{\alpha_0} - z) \rangle - \langle x_{\alpha_0} - y, J(y - z) \rangle| \\ & \leq |\langle x_{\alpha_0} - y_{\alpha_0}, J(y_{\alpha_0} - z) \rangle - \langle x_{\alpha_0} - y, J(y_{\alpha_0} - z) \rangle| \\ & \quad + |\langle x_{\alpha_0} - y, J(y_{\alpha_0} - z) \rangle - \langle x_{\alpha_0} - y, J(y - z) \rangle| \\ & = |\langle y - y_{\alpha_0}, J(y_{\alpha_0} - z) \rangle| + |\langle x_{\alpha_0} - y, J(y_{\alpha_0} - z) - J(y - z) \rangle| \\ & \leq \|y - y_{\alpha_0}\| \|y_{\alpha_0} - z\| + \|x_{\alpha_0} - y\| \|J(y_{\alpha_0} - z) - J(y - z)\| \\ & < \delta_0 K + \delta K < \varepsilon \end{aligned}$$

and hence

$$\langle x_{\alpha_0} - y, J(y - z) \rangle > \langle x_{\alpha_0} - y_{\alpha_0}, J(y_{\alpha_0} - z) \rangle - \varepsilon \geq -\varepsilon.$$

From $z \in \overline{\text{co}}\{x_{\alpha_0} : \alpha \in S\}$, we have

$$\langle z - y, J(y - z) \rangle \geq -\varepsilon,$$

that is

$$\|y - z\|^2 \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $y = z$. \square

LEMMA 3.8 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. Let $x \in C$. Assume that $F(S) \neq \emptyset$. Then for $y \in F(S)$ and $y \notin Q(x)$,*

$$k = \inf_s \|T_s x - y\| > 0.$$

Proof. Supposing that $k = 0$, by Lemma 3.5,

$$\inf_s \sup_t \|T_{t,s} x - y\| = k = 0.$$

Let $z \in Q(x)$. For each $t \in S$, let y_t be the unique element in $[y, T_t x]$ such that

$$\|y_t - z\| = \min\{\|u - z\| : u \in [y, T_t x]\}.$$

So, for any $\varepsilon > 0$, there exists $s_0 \in S$ such that

$$\sup_t \|T_{ts_0} x - y\| < \frac{\varepsilon}{2}$$

and hence we have

$$\begin{aligned} \|y_{ts_0} - y\| &\leq \|y_{ts_0} - T_{ts_0} x\| + \|T_{ts_0} x - y\| \\ &\leq 2\|T_{ts_0} x - y\| < \varepsilon \end{aligned}$$

for every $t \in S$, that is, $y_t \rightarrow y$ ($t \rightarrow \infty_R$). So by Lemma 3.7, we have $y = z$. This is a contradiction. So we have $k > 0$. \square

LEMMA 3.9 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. Let $x \in C$. Then for any $y \in F(S)$ and $z \in Q(x)$, there exists a closed left ideal L of S such that*

$$\langle T_t x - y, J(y - z) \rangle \leq 0$$

for every $t \in L$.

Proof. If $x = y$ or $y = z$, Lemma 3.9 is obvious. So, let $x \neq y$ and $y \neq z$. For any $t \in S$, define a unique element y_t such that $y_t \in [y, T_t x]$ and

$$\|y_t - z\| = \min\{\|u - z\| : u \in [y, T_t x]\}.$$

Then since $y \neq z$, by Lemma 3.7 we have $y_t \not\rightarrow y$ ($t \rightarrow \infty_R$). So we obtain $c > 0$ such that for any $t \in S$, there exists $t' \in S$ with $\|y_{t't} - y\| \geq c$. Setting

$$y_{t't} = a_{t't} T_{t't} x + (1 - a_{t't})y, \quad a_{t't} \in [0, 1],$$

we also obtain $c_0 > 0$ so small that $a_{t't} \geq c_0$. In fact, since $T_{t't}$ is nonexpansive and $y \in F(S)$, we have

$$c \leq \|y_{t't} - y\| = a_{t't} \|T_{t't} x - y\| \leq a_{t't} \|x - y\|.$$

So, put $c_0 = c/\|x - y\|$. Let $k = \inf_s \|T_s x - y\|$. By Lemma 3.5 and $y_t \not\rightarrow y$ ($t \rightarrow \infty_R$), we have $k > 0$.

Now, choose $\varepsilon > 0$ so small that

$$(R + \varepsilon) \left(1 - \delta \left(\frac{c_0 k}{R + \varepsilon} \right) \right) < R,$$

where δ is the modulus of convexity of E and $R = \|z - y\|$. Then by Lemma 3.6, there exists $t_0 \in S$ such that

$$\|T_s(c_0 T_{t_0} x + (1 - c_0)y) - (c_0 T_s T_{t_0} x + (1 - c_0)y)\| < \varepsilon \quad (*)$$

for every $s, t \in S$. Fix $t_1 \in S$ with $\|y_{t_1 t_0} - y\| \geq c$. Then since $a_{t_1 t_0} \geq c_0$, we have

$$\begin{aligned} c_0 T_{t_1 t_0} x + (1 - c_0) y &= \left(1 - \frac{c_0}{a_{t_1 t_0}}\right) y + \frac{c_0}{a_{t_1 t_0}} (a_{t_1 t_0} T_{t_1 t_0} x + (1 - a_{t_1 t_0}) y) \\ &= \left(1 - \frac{c_0}{a_{t_1 t_0}}\right) y + \frac{c_0}{a_{t_1 t_0}} y_{t_1 t_0} \in [y, y_{t_1 t_0}] \end{aligned}$$

and hence

$$\begin{aligned} \|c_0 T_{t_1 t_0} x + (1 - c_0) y - z\| &\leq \max\{\|y - z\|, \|y_{t_1 t_0} - z\|\} \\ &\leq \|y - z\| = R. \end{aligned}$$

By using (*), we obtain

$$\begin{aligned} \|c_0 T_s T_{t_1 t_0} x + (1 - c_0) y - z\| &< \|T_s(c_0 T_{t_1 t_0} x + (1 - c_0) y) - z\| + \varepsilon \\ &\leq \|c_0 T_{t_1 t_0} x + (1 - c_0) y - z\| + \varepsilon \\ &\leq R + \varepsilon \end{aligned}$$

for every $s \in S$. On the other hand, since $\|y - z\| = R < R + \varepsilon$ and

$$\|c_0 T_s T_{t_1 t_0} x + (1 - c_0) y - y\| = c_0 \|T_{s t_1 t_0} x - y\| \geq c_0 k$$

for every $s \in S$, we have, by uniform convexity,

$$\begin{aligned} &\left\| \frac{1}{2} ((c_0 T_s T_{t_1 t_0} x + (1 - c_0) y - z) + (y - z)) \right\| \\ &\leq (R + \varepsilon) \left(1 - \delta \left(\frac{c_0 k}{R + \varepsilon}\right)\right) < R, \end{aligned}$$

that is

$$\left\| \frac{c_0}{2} T_s T_{t_1 t_0} x + \left(1 - \frac{c_0}{2}\right) y - z \right\| < R$$

for every $s \in S$. Putting

$$u_s = \frac{c_0}{2} T_s T_{t_1 t_0} x + \left(1 - \frac{c_0}{2}\right) y,$$

we have

$$\begin{aligned} \|u_s + \alpha(y - u_s) - z\| &= \|\alpha(y - z) - (\alpha - 1)(u_s - z)\| \\ &\geq \alpha\|y - z\| - (\alpha - 1)\|u_s - z\| \\ &\geq \alpha\|y - z\| - (\alpha - 1)\|y - z\| = \|y - z\| \end{aligned}$$

for every $s \in S$ and $\alpha \geq 1$. So, by Theorem 2.5 in [7], we have

$$\langle u_s + \alpha(y - u_s) - y, J(y - z) \rangle \geq 0$$

for every $s \in S$ and $\alpha \geq 1$ and hence

$$\langle u_s - y, J(y - z) \rangle \leq 0$$

for every $s \in S$. Therefore we obtain

$$\begin{aligned} \langle T_s T_{t_1 t_0} x - y, J(y - z) \rangle &= \frac{2}{c_0} \left\langle \frac{c_0}{2} T_s T_{t_1 t_0} x - \frac{c_0}{2} y, J(y - z) \right\rangle \\ &= \frac{2}{c_0} \langle u_s - y, J(y - z) \rangle \leq 0 \end{aligned}$$

for every $s \in S$. Let $L = \overline{S t_1 t_0}$. \square

LEMMA 3.10 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $x \in C$. If for any $y, z \in Q(x) \cap F(S)$,*

$$\inf_{L \in \mathcal{L}(S)} \inf_{\phi \in J(y-z)} \sup_{t \in L} \langle T_t x - y, \phi \rangle \leq 0,$$

then $Q(x) \cap F(S)$ has at most one point.

Proof. Let $y, z \in Q(x) \cap F(S)$. Then by convexity of $Q(x) \cap F(S)$, we have $(y+z)/2 \in Q(x) \cap F(S)$. Let $\varepsilon > 0$. By assumption, there exist $L \in \mathcal{L}(S)$ and $\phi \in J((y+z)/2 - z)$ such that

$$\left\langle T_t x - \frac{y+z}{2}, \phi \right\rangle \leq \varepsilon$$

for every $t \in L$. Since $y \in \overline{\text{co}}\{T_t x : t \in L\}$, it follows

$$\left\langle y - \frac{y+z}{2}, \phi \right\rangle \leq \varepsilon$$

and hence

$$\frac{1}{2} \langle y - z, \phi \rangle = \frac{1}{2} \|y - z\|^2 \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $y = z$. \square

Combining Lemma 3.9 and Lemma 3.10, we have the following result.

THEOREM 3.11 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. Let $x \in C$. Then $Q(x) \cap F(S)$ contains at most one point.*

4 Ergodic theorems

We are now ready to prove our main nonlinear ergodic theorems.

THEOREM 4.1 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. Let $S = \{T_s : s \in S\}$ be a continuous representation of a semitopological semigroup S as nonexpansive mappings from C into C . Assume that $F(S) \neq \emptyset$. Then the following are equivalent:*

- (1) *For each $x \in C$, the set $Q(x) \cap F(S)$ is nonempty.*

(2) There exists a retraction P of C onto $F(S)$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{\text{co}}\{T_tx : t \in S\}$ for every $x \in C$.

Proof. (1) \Rightarrow (2). If for each $x \in C$, the set $Q(x) \cap F(S) \neq \emptyset$, then by Theorem 3.11, $Q(x) \cap F(S)$ contains exactly one point Px . Then clearly P is a retraction of C onto $F(S)$ and $Px \in \overline{\text{co}}\{T_tx : t \in S\}$ for every $x \in C$. Clearly $T_tP = P$ for every $t \in S$. Also if $u \in S$ and $x \in C$, we have

$$\bigcap_{s \in S} \overline{\text{co}}\{T_{ts}x : t \in S\} \subset \bigcap_{s \in S} \overline{\text{co}}\{T_{tsu}x : t \in S\}$$

and hence

$$Q(x) \cap F(S) = Q(T_u x) \cap F(S).$$

This implies $PT_t = P$ for every $t \in S$.

(2) \Rightarrow (1). Let $x \in C$. Then it is obvious that $Px \in F(S)$. Since

$$Px = PT_s x \in \overline{\text{co}}\{T_t T_s x : t \in S\} = \overline{\text{co}}\{T_{ts}x : t \in S\}$$

for every $s \in S$, we have

$$Px \in \bigcap_{s \in S} \overline{\text{co}}\{T_{ts}x : t \in S\} = Q(x). \square$$

THEOREM 4.2 Let C be a nonempty closed convex subset of a Hilbert space H and let $S = \{T_s : s \in S\}$ be a continuous representation of a semitopological semigroup S as nonexpansive mappings from C into C . If for each $x \in C$, the set $Q(x) \cap F(S)$ is nonempty, then there exists a nonexpansive retraction P of C onto $F(S)$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{\text{co}}\{T_tx : t \in S\}$ for every $x \in C$.

Proof. For each $x \in C$, let Px be the unique element in $Q(x) \cap F(S)$. Then, as in the proof of Theorem 4.1 (1) \Rightarrow (2), P is a retraction of C onto $F(S)$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{\text{co}}\{T_tx : t \in S\}$ for every $x \in C$. It remains to show that P is nonexpansive. Let $y \in C$ and $0 < \lambda < 1$. Then as in the proof of Theorem 3.3 we have for any $\varepsilon > 0$,

$$\begin{aligned} & q_x((1 - \lambda)Px + \lambda Py) \\ &= \sup_s \inf_t \|T_{ts}x - ((1 - \lambda)Px + \lambda Py)\|^2 \\ &= \sup_s \inf_t \|T_{ts}x - Px + \lambda(Px - Py)\|^2 \\ &= \sup_s \inf_t (\|T_{ts}x - Px\|^2 + 2\lambda \langle T_{ts}x - Px, Px - Py \rangle + \lambda^2 \|Px - Py\|^2) \\ &< q_x(Px) + 2\lambda \sup_s \inf_t \langle T_{ts}x - Px, Px - Py \rangle + \lambda^2 \|Px - Py\|^2 + \varepsilon. \end{aligned}$$

Since Px is the minimizer of q_x , we have

$$\begin{aligned} & 2\lambda \sup_s \inf_t \langle T_{ts}x - Px, Px - Py \rangle + \lambda^2 \|Px - Py\|^2 + \varepsilon \\ & > q_x((1 - \lambda)Px + \lambda Py) - q_x(Px) \geq 0. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$2\lambda \sup_s \inf_t \langle T_{ts}x - Px, Px - Py \rangle + \lambda^2 \|Px - Py\|^2 \geq 0$$

and hence

$$2 \sup_s \inf_t \langle T_{ts}x - Px, Px - Py \rangle \geq -\lambda \|Px - Py\|^2.$$

Now, if $\lambda \rightarrow 0$, then

$$\sup_s \inf_t \langle T_{ts}x - Px, Px - Py \rangle \geq 0.$$

Let $\varepsilon > 0$. Then there exists $u \in S$ such that

$$\langle T_{tu}x - Px, Px - Py \rangle > -\varepsilon$$

for every $t \in S$. For such an element $u \in S$, we also have

$$\sup_s \inf_t \langle T_{ts}T_u y - PT_u y, PT_u y - Px \rangle \geq 0$$

and hence there exists $v \in S$ such that

$$\langle T_{tvu}y - PT_u y, PT_u y - Px \rangle > -\varepsilon$$

for every $t \in S$. Then, from $PT_u y = Py$, we have

$$\langle T_{tvu}y - Py, Py - Px \rangle > -\varepsilon$$

for every $t \in S$. Therefore we have

$$\begin{aligned} -2\varepsilon &< \langle T_{uvu}x - Px, Px - Py \rangle + \langle T_{uvu}y - Py, Py - Px \rangle \\ &= \langle T_{uvu}x - T_{uvu}y - (Px - Py), Px - Py \rangle \\ &= \langle T_{uvu}x - T_{uvu}y, Px - Py \rangle - \|Px - Py\|^2 \\ &\leq \|T_{uvu}x - T_{uvu}y\| \|Px - Py\| - \|Px - Py\|^2 \\ &\leq \|x - y\| \|Px - Py\| - \|Px - Py\|^2. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this implies $\|Px - Py\| \leq \|x - y\|$. \square

We now proceed to find conditions on S and E such that $Q(x) \cap F(S) \neq \emptyset$ for every $x \in C$.

LEMMA 4.3 [20] *Let C be a nonempty closed convex subset of a Hilbert space H , let S be an index set, and let $\{x_t : t \in S\}$ be a bounded set of H . Let X be a subspace of $l^\infty(S)$ containing constants, and let μ be a submean on X . Suppose that for each $x \in C$, the real-valued function f on S defined by*

$$f(t) = \|x_t - x\|^2 \text{ for all } t \in S$$

belongs to X . If

$$r(x) = \mu_t \|x_t - x\|^2 \text{ for all } x \in C$$

and $r = \inf\{r(x) : x \in C\}$, then there exists a unique element $z \in C$ such that $r(z) = r$. Further the following inequality holds :

$$r + \|z - x\|^2 \leq r(x) \text{ for every } x \in C.$$

THEOREM 4.4 *Let C be a nonempty closed convex subset of a Hilbert space H and let S be a semitopological semigroup such that $RUC(S)$ has an invariant submean. Let $S = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings from C into C . Suppose that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Then the set $Q(x) \cap F(S)$ is nonempty.*

Proof. First we observe that for any $y \in H$, the function $f(t) = \|T_t x - y\|^2$ is in $RUC(S)$ (see [16]). Let μ be an invariant submean and define a real-valued function g on H by

$$g(y) = \mu_t \|T_t x - y\|^2 \text{ for each } y \in H.$$

If $r = \inf\{g(y) : y \in H\}$, then by Lemma 4.3 there exists a unique element $z \in H$ such that $g(z) = r$. Further, we know that

$$r + \|z - y\|^2 \leq g(y) \text{ for every } y \in H.$$

For each $s \in S$, let Q_s be the metric projection of H onto $\overline{\text{co}}\{T_{ts}x : t \in S\}$. Then by Phelps [22], Q_s is nonexpansive and for each $t \in S$,

$$\|T_{ts}x - Q_s z\|^2 = \|Q_s T_{ts}x - Q_s z\|^2 \leq \|T_{ts}x - z\|^2.$$

So, we have

$$\begin{aligned} \mu_t \|T_t x - Q_s z\|^2 &= \mu_t \|T_{ts}x - Q_s z\|^2 \\ &\leq \mu_t \|T_{ts}x - z\|^2 \\ &= \mu_t \|T_t x - z\|^2 \end{aligned}$$

and thus $Q_s z = z$. This implies

$$z \in \overline{\text{co}}\{T_{ts}x : t \in S\} \text{ for all } s \in S$$

and hence

$$z \in \bigcap_{s \in S} \overline{\text{co}}\{T_{ts}x : t \in S\}.$$

On the other hand, by Lemma 4.3

$$\|z - y\|^2 \leq \mu_t \|T_t x - y\|^2 - \mu_t \|T_t x - z\|^2 \text{ for every } y \in H.$$

So, putting $y = T_s z$ for each $s \in S$, we have

$$\begin{aligned} \|z - T_s z\|^2 &\leq \mu_t \|T_t x - T_s z\|^2 - \mu_t \|T_t x - z\|^2 \\ &= \mu_t \|T_{st}x - T_s z\|^2 - \mu_t \|T_t x - z\|^2 \\ &\leq \mu_t \|T_t x - z\|^2 - \mu_t \|T_t x - z\|^2 = 0. \end{aligned}$$

Therefore, we have $T_s z = z$ for every $s \in S$. \square

PROPOSITION 4.5 *Let S be a discrete reversible semigroup and let C be a nonempty weakly compact convex subset of a Banach space E . Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as affine nonexpansive mappings from C into C . Then for each $x \in C$, the set $Q(x) \cap F(\mathcal{S})$ is nonempty.*

Proof. Let $x \in C$. Clearly $Q(x) = \bigcap_{s \in S} \overline{\text{co}}\{T_{ts}x : t \in S\}$ is compact and convex. Also $Q(x)$ is nonempty. Indeed, for each $s \in S$, let $K_s = \overline{\text{co}}\{T_{ts}x : t \in S\}$. Then $\{K_s : s \in S\}$ are weakly closed subsets of C with finite intersection property: If $s_1, \dots, s_n \in S$, choose $t_0 \in \bigcap_{i=1}^n Ss_i$. Then $T_{t_0}x \in \bigcap_{i=1}^n K_{s_i}$. Consequently $\bigcap_{s \in S} K_s = Q(x)$ is nonempty. Let $a, s \in S, y \in Q(x)$ and let $\{y_k\}$ be a sequence in K_s such that $\|y_k - y\| \rightarrow 0$. Then $T_a y_k \in K_s$ (by affinity and continuity of T_a) and $\|T_a y_k - T_a y\| \leq \|y_k - y\| \rightarrow 0$. Hence $T_a y \in K_s$. Consequently $T_a y \in \bigcap_{s \in S} K_s = Q(x)$. Hence $Q(x)$ is \mathcal{S} -invariant. Now since S is left reversible, by [14] the space $WAP(S)$ of weakly almost periodic functions on S has a left invariant mean (see also [17]). Hence by [15], there exists a common fixed point in $Q(x)$, i.e., $Q(x) \cap F(\mathcal{S}) \neq \emptyset$. \square

5 Minimal left ideals

Let (Σ, \circ) be a compact right topological semigroup, i.e., a smigroup and a compact Hausdorff topological space such that for each $\tau \in \Sigma$ the mapping $\gamma \rightarrow \gamma \circ \tau$ from Σ into Σ is continuous. In this case, Σ must contain minimal left ideals. Any minimal left ideal in Σ is closed and any two minimal left ideals of Σ are homeomorphic and algebraically isomorphic (see [3]).

LEMMA 5.1 *Let X be a nonempty weakly compact convex subset of a Banach space E . Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of a semigroup S as nonexpansive and weak-weak continuous mappings from X into X . Let Σ be the closure of \mathcal{S} in the product space $(X, \text{weak})^X$. Then Σ is a compact right topological semigroup consisting of nonexpansive mappings from X into X . Further, for any $T \in \Sigma$, there exists a sequence $\{T_n\}$ of convex combination of operators from \mathcal{S} such that $\|T_n x - Tx\| \rightarrow 0$ for every $x \in X$.*

Proof. It is easy to see that Σ is a compact right topological semigroup. We now prove the last statement (which implies that each $T \in \Sigma$ is nonexpansive). Consider $\mathcal{S} \subseteq (E, \|\cdot\|)^X$ with the product topology. Let $\Phi = \text{co}\mathcal{S}$. Then each $T \in \Phi$ is nonexpansive. Hence each $T \in \overline{\Phi}$ is also nonexpansive. Since the weak topology of the locally convex space $(E, \|\cdot\|)^X$ is the product space $(E, \text{weak})^X$, it follows that $\Sigma \subseteq \overline{\Phi}^{\text{weak}} = \overline{\Phi}$, and hence the last statement holds. \square

Σ is called the enveloping semigroup of \mathcal{S} .

A subset X of a Banach space E is said to have normal structure if for any bounded (closed) convex subset W of X which contains more than one point, there exists $x \in W$ such that $\sup\{\|x - y\| : y \in W\} < \text{diam}(W)$, where $\text{diam}(W) = \sup\{\|x - y\| : x, y \in W\}$ (see [10] for more details).

THEOREM 5.2 *Let X be a nonempty weakly compact convex subset of a Banach space E and X has normal structure. Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of a semigroup*

as norm-nonexpansive and weakly continuous mappings from X into X and let Σ be the enveloping of S . Let I be a minimal left ideal of Σ and let Y be a minimal S -invariant closed convex subset of X . Then there exists a nonempty weakly closed subset C of Y such that I is constant on C .

Proof. Since I is a minimal left ideal of Σ and Σ is a compact right topological semigroup (Lemma 5.1), $I = \Sigma e$ for a minimal idempotent e of Σ and $G = e\Sigma e$ is a maximal subgroup contained in I (see [3]). Since each $T \in G$ is a nonexpansive mapping from Y into Y (Lemma 5.1), by Broskii-Milman Theorem [4], there exists $x \in Y$ such that $Tx = x$ for every $T \in G$. Now put $C = Ix$. Then C is weakly closed and S -invariant. Also if $y_1, y_2 \in C, y_1 = T_1ex, y_2 = T_2ex, T_1, T_2 \in \Sigma$, then, since $eT_1e \in G$, we have

$$(Te)y_1 = Te(T_1ex) = Tx$$

for every $T \in \Sigma$ and similarly

$$(Te)y_2 = Tx$$

for every $T \in \Sigma$. The assertion is proved. \square

The following improves the main theorem in [13] for Banach spaces (see also [21]).

COROLLARY 5.3 *Let Σ and X be as in Theorem 5.2. Then there exist $T_0 \in \Sigma$ and $x \in X$ such that $T_0Tx = T_0x$ for every $T \in \Sigma$.*

Proof. Pick $x \in C$ and $T_0 \in I$ of the above theorem. \square

REMARK 5.4 *If S is commutative, then for any $T \in \Sigma$ and $s \in S, T_s \circ T = T \circ T_s$, i.e., $z = T_0x$ is in fact a common fixed point for Σ (and hence for S). Note that if X is norm compact, the weak and norm topology agree on X . Hence every nonexpansive mapping from X into X must be weakly continuous. Therefore Corollary 5.3 improves the well known fixed point theorem of De Marr [6] for commuting semigroups of nonexpansive mappings on compact convex sets.*

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